

# ON THE RATIONAL HOMOLOGY AND ASSEMBLY MAPS OF GENERALISED THOMPSON GROUPS

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**ABSTRACT.** Let  $V_r(\Sigma)$  be the generalised Thompson group defined as the automorphism group of a valid, bounded, and complete Cantor algebra. We show that the rational homology of  $V_r(\Sigma)$  vanishes above a certain degree depending only on  $s$ , the number of descending operations used in the definition of the Cantor algebra. In particular this applies to Brin-Thompson groups  $sV$ . We explicitly compute the rational homology for  $2V$  and show that it vanishes everywhere except in degrees 0 and 3, where it is isomorphic to  $\mathbb{Q}$ . We also determine the number of conjugacy classes of finite cyclic subgroups of a given order  $m$  in Brin-Thompson groups. We apply our computations to the rationalised Farrell-Jones assembly map in algebraic  $K$ -theory.

## 1. INTRODUCTION

There are many well-known generalisations of Thompson's group  $V$  due to, for example, Higman [10], Stein [16], and Brin [2], which share many of the properties of  $V$ . For example, they all contain infinite torsion and are of type  $F_\infty$  [4, 8]. Furthermore, the Higman-Thompson groups  $V_{n,r}$  and the Brin-Thompson groups  $sV$  are simple [10, 3]. It turns out that all these are examples of automorphism groups  $V_r(\Sigma)$  of valid, bounded, and complete Cantor algebras. For notation and definitions we refer to [13, 14], where it was shown that these  $V_r(\Sigma)$  and the centralisers of their finite subgroups are of type  $F_\infty$ , and an explicit description of these centralisers was given.

Brown showed in [5] that Thompson's group  $V$  is rationally acyclic; he also conjectured that  $V$  is acyclic, and this conjecture was recently proved in [17]. In this paper we show that the rational homology of  $V_r(\Sigma)$  vanishes above a certain degree depending only on the number  $s$  of descending operations used in the definition of the Cantor algebra. In particular:

**Theorem 1.1.** *Let  $U_r(\Sigma)$  be a complete, valid, and bounded Cantor algebra,  $s \geq 2$  and  $k \geq s2^{s-1}$ . Then  $H_k(V_r(\Sigma); \mathbb{Q}) = 0$ .*

This shows in particular that all Higman-Thompson groups  $V_{n,r}$  are rationally acyclic.

The method of the proof gives a very explicit construction to actually compute the homology groups, which we do for Brin's group  $V_r(\Sigma) = 2V$ . It turns out that its rational homology vanishes everywhere except in degrees 0 and 3, where it is isomorphic to  $\mathbb{Q}$ ; compare Theorem 2.11.

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In [13] it was shown that, for any finite subgroup  $K \leq V_r(\Sigma)$ , there are only finitely many conjugacy classes of subgroups isomorphic to  $K$ . Using the method employed there, in Proposition 3.1 we explicitly calculate the number of conjugacy classes of cyclic subgroups of a given order in the Brin-like groups  $sV_{n,r}$ , where  $n$  denotes the arity of the descending operations. Note that  $sV = sV_{2,1}$ .

In Section 4 we discuss an application of our results to the algebraic  $K$ -theory of the integral group rings of these generalized Thompson groups. We first review the Farrell-Jones Conjecture, and then explain how our computations, combined with the results from [12, 13, 14], imply the following theorem. This generalizes the analysis that was carried out for Thompson's group  $T$  in [?]. All terms and notation are explained in Section 4.

**Theorem 1.2.** *Consider a Brin-Thompson group  $G = sV$ .*

*If the rationalised Farrell-Jones Conjecture in algebraic  $K$ -theory holds for  $sV$ , then  $K_n(\mathbb{Z}[sV]) \otimes_{\mathbb{Z}} \mathbb{Q}$  is isomorphic to*

$$(1.3) \quad \bigoplus_{1 \leq m} \bigoplus_{1 \leq i \leq 2^{\phi(m)} - 1} \bigoplus_{\substack{p+q=n \\ 0 \leq p \leq \phi(m)s2^{s-1} \\ -1 \leq q}} H_p(Z_G(C_m^i); \mathbb{Q}) \otimes_{\mathbb{Q}[W_G(C_m^i)]} \Theta_{C_m^i} \left( K_q(\mathbb{Z}[C_m^i]) \otimes_{\mathbb{Z}} \mathbb{Q} \right),$$

where  $C_m^i$  denote the representatives of the conjugacy classes of finite cyclic groups of order  $m$  in  $sV$ .

*If the Leopoldt-Schneider Conjecture holds for all cyclotomic fields, then  $K_n(\mathbb{Z}[sV]) \otimes_{\mathbb{Z}} \mathbb{Q}$  contains as a direct summand the subspace of (1.3) indexed by  $q \geq 0$ .*

The results used in the proof of this Theorem also imply that the homology groups  $H_p(Z_G(C_m^i))$  can be computed in terms of the ordinary homology  $H_p(sV)$  of  $sV$ . Finally, we remark that using [12, 14] one also deduces that for any valid, bounded, and complete Cantor algebra, the rationalised Whitehead group  $Wh(V_r(\Sigma)) \otimes_{\mathbb{Z}} \mathbb{Q}$  is an infinite dimensional  $\mathbb{Q}$ -vector space; compare Corollary 4.5.

## 2. THE RATIONAL HOMOLOGY OF $V_r(\Sigma)$

Throughout this section we will use the notation used in [14]. Let  $U_r(\Sigma)$  denote a valid, bounded, and complete Cantor algebra, and let  $G = V_r(\Sigma)$  be its automorphism group. The examples to keep in mind are those where  $r = 1$ , and  $G = V_1(\Sigma)$  is either the Higman-Thompson-group  $G = V_{n,1}$  or the Brin-Thompson group  $sV$ . For each basis  $A \in U_r(\Sigma)$  we have a Morse function  $t(A) = |A|$ . We denote the Stein complex by  $X$ , which was denoted by  $\mathcal{S}_r(\Sigma)$  in [14].

*Remark 2.1.* The Stein complex is known to be contractible [5, 8, 14]. By the same argument as in the remark at the end of Section 4 of [5], nothing changes if one fixes an integer  $q$  and replaces the Stein complex by the Stein complex obtained by only considering bases of cardinality  $\geq q$ .

**The truncated Stein complex.** As in [5] we will consider a truncated Stein complex. For  $1 \leq p \leq q$  let  $X_{p,q}$  denote the full subcomplex of  $X$  generated by bases  $A$  with  $p \leq |A| \leq q$ . Note that  $\dim X_{p,q} \leq q - p$ .

Recall from [14, Lemma 3.4] that each basis has a unique maximal elementary descendant, denoted  $\mathcal{E}(A)$ , with  $n_1 \cdots n_s |A|$  leaves obtained by applying all descending operations exactly once to each element of  $A$ . Hence  $\dim X_{p,q} = q - p$  if  $q \leq Np$ , where  $N = n_1 \cdots n_s$ .

*Remark 2.2.* For  $V$  we have  $N = 2$ ; for  $V_n$  we have  $N = n$ ; and for  $sV$  we have  $N = 2^s$ .

**Proposition 2.3.** *Suppose that  $s \geq 2$  and let  $U_r(\Sigma)$  be a complete, valid, and bounded Cantor algebra. For all  $n \geq 1$  there is an integer  $p_0$ , depending on  $n$ , such that  $X_{p,q}$  is  $n$ -connected for  $p \geq p_0$  and  $q \geq p + (N - 1)n$ .*

*Proof.* Let  $t = N - 1$ . Since the complex given by the union of

$$X_{p,p+tn} \subset X_{p,p+tn+1} \subset \cdots$$

is contractible, see Remark 2.1, it suffices to show that there is some  $p_0$  such that for  $p \geq p_0$  and  $q \geq p + tn$ , the pair  $(X_{p,q+1}, X_{p,q})$  is  $n$ -connected, i.e., the inclusion  $X_{p,q} \hookrightarrow X_{p,q+1}$  induces an isomorphism between the homotopy groups  $\pi_i$  for  $0 \leq i \leq n$ . Exactly as in [5, Theorem 2], this will be satisfied if for any  $A$  with  $|A| = q + 1$ , then

$$L^p(A) = |\{B \in X_{p,q} \mid B < A\}|$$

is  $n$ -connected. We follow the lines of the argument of [8], and begin by showing that

$$L_0^p(A) = |\{B \in X_{p,q} \mid B < A \text{ very elementary}\}|$$

is  $n$ -connected. Consider the complex  $K_q$ , which was denoted  $K_n$  in the proof of [14, Lemma 3.8]. This complex has as vertices the labelled subsets of  $A$  where the possible labels are precisely the set of colours and the cardinality of a subset with label  $i$  is the arity  $n_i$ . The simplices are given by pairwise disjoint set of such subsets. As in [14, Lemma 3.8], the argument of [4, Lemma 4.20] shows that  $K_q$  is  $n$ -connected if  $q$  is big enough. Now,  $q - p - 1 \geq tn - 1 \geq n + 1$  and therefore the  $(q - p - 1)$ -skeleton of  $K_q$  is also  $n$ -connected. Finally observe that  $\dim L_p(A) = q - p - 1$  and in fact,  $L_p(A)$  is the barycentric subdivision of that  $(q - p - 1)$ -skeleton.

Now let  $B \in L^p(A) \setminus L_0^p(A)$  and consider its descending link  $\text{lk}^p \downarrow (B)$  with respect to the height function  $\bar{h}$ , but now truncating above  $p$ . We claim that this link is  $(n - 1)$ -connected. Using Morse theory (see [8, Lemma 3.1]) this will yield the result. We have

$$\text{lk}^p \downarrow (B) = \text{downlk}^p \downarrow (B) * \text{uplk}^p \downarrow (B),$$

where

$$\text{downlk}^p \downarrow (B) = |\{C \mid C < B, \bar{h}(C) < \bar{h}(B), p \leq |C|\}|$$

is the truncated downlink, and

$$\text{uplk}^p \downarrow (B) = \text{uplk} \downarrow (B) = |\{C \mid B < C, \bar{h}(C) < \bar{h}(B)\}|$$

is the uplink (truncation does not affect the uplink).

As in [8], we may assume that no element of  $A$  is obtained by a simple expansion of an element in  $B$ , otherwise the uplink is contractible and the claim holds true. This means that each element of  $B$  yields either some number  $l$  with  $\min\{n_i \mid i \in S\} + 1 = m + 1 \leq l \leq N$  of elements in  $A$ , or

remains invariant in  $A$ . Let  $k_{b,l}$  be the number of elements of  $B$  yielding exactly  $l$  elements in  $A$  and let  $k_s$  be the number of the invariant elements of  $B$ . Put  $k = \sum_{l=m}^N k_{b,l}$ . Then

$$(2.4) \quad |B| = k + k_s,$$

and

$$(2.5) \quad |A| = \sum_{l=m}^N l k_{b,l} + k_s = q + 1 \geq p + tn + 1.$$

Set  $|B| = x$ . Now,  $\text{downlk}^p \downarrow (B)$  is equivalent to the complex associated to

$$L_0^{x-p}(A_1),$$

where  $A_1$  has exactly  $k_s$  elements. Note that this complex has dimension  $x - p$ . Therefore, choosing  $k_s$  big enough, we may assume that the downlink is  $(x - p - 2)$ -connected.

On the other hand, the argument of [8] implies that the uplink is  $(k - 2)$ -connected. Therefore,  $\text{lk}^p \downarrow (B)$  is  $(x - p - 2 + k)$ -connected.

Now, from (2.4) and (2.5) we get

$$tk + x \geq \sum_{l=m}^N (l - 1) k_{b,l} + x \geq p + tn + 1.$$

Thus

$$x - p - 1 - t + tk \geq tn - t.$$

Assume first that  $x - p \neq 0$ . Then  $t(x - p - 2) \geq x - p - 1 - t$ , and hence

$$t(x - p - 2 + k) \geq x - p - 1 - t + tk \geq tn - t.$$

Finally,  $x - p - 2 + k \geq n - 1$ . In the case when  $p - x = 0$  we have

$$tk - 1 \geq tn.$$

Thus  $k - 1/t \geq n$ , and since both  $k$  and  $n$  are integers, this implies that  $k - 1 \geq n$ , and therefore

$$k - 2 \geq n - 1.$$

This means that in both cases  $\text{lk}^p \downarrow (B)$  is  $(n - 1)$ -connected, as required.  $\square$

From now on fix a basis  $A$  with  $|A| = p$ , and let  $Y_{p,q} = X_{p,q}/G$ . Consider the complex  $\widehat{Z}_{p+1,q}$  with  $m$ -simplices of the form  $\sigma : B_0 < B_1 < \dots < B_m$  with  $|B_m| \leq p + n$ ,  $A < B_0$  and  $A \leq B_m$  elementary. We let the (finite) group

$$H = \{g \in G \mid gA = A\}$$

act on  $\widehat{Z}_{p+1,q}$ , and set

$$Z_{p+1,q} = \widehat{Z}_{p+1,q}/H.$$

**Lemma 2.6.**  *$Y_{p+1,q}$  is a subcomplex of  $Y_{p,q}$ , and there is an isomorphism of chain complexes*

$$\mathcal{C}_*(Y_{p,q}, Y_{p+1,q}) \cong \mathcal{C}_*(CZ_{p+1,q}, Z_{p+1,q})$$

(where  $C$  denotes the cone) and therefore a long exact sequence in homology

$$\dots \rightarrow H_j(Y_{p+1,q}) \rightarrow H_j(Y_{p,q}) \rightarrow H_j(CZ_{p+1,q}, Z_{p+1,q}) \cong \tilde{H}_{j-1}(Z_{p+1,q}) \rightarrow \dots$$

*Proof.* Note first that  $CZ_{p+1,q}$  can be seen as the complex with  $m$ -simplices of the form  $H(B_0 < B_1 < \cdots < B_m)$  with  $A \leq B_0$ . The fact that  $Y_{p+1,q}$  lies inside  $Y_{p,q}$  is obvious. Moreover, if  $G\sigma$  is an orbit with  $G\sigma \in Y_{p,q} \setminus Y_{p+1,q}$ , then  $\sigma$  has the form  $\sigma : A_0 < A_1 < \cdots < A_m$  with  $|A_0| = p$ . We may choose some  $g \in G$  with  $gA_0 = A$ , and map

$$\varphi : G\bar{\sigma} \mapsto H(\bar{\nu}),$$

where  $\nu = A < gA_1 < \cdots < gA_m$ , and  $\bar{\phantom{x}}$  denotes passing to the quotient chain complex. Note that if we choose some other  $g' \neq g$  with  $g'A_0 = A$ , then  $g = hg'$  for some  $h \in H$ , thus

$$H(A < gA_1 < \cdots < gA_m) = H(A < g'A_1 < \cdots < g'A_m).$$

In particular this map does not depend on the choice of  $g$ . It is obvious that  $\varphi$  induces an isomorphism at each degree of the chain complexes, we only have to show that this is in fact a chain map. But this follows from the fact that, if  $\delta$  is the boundary map, then

$$\delta(G\bar{\sigma}) = \sum_{i=1}^m (-1)^i G(\bar{\tau}_i)$$

with  $\tau_i = A_0 < \cdots < A_{i-1} < A_{i+1} < \cdots < A_m$ . □

The main advantage of this result is that  $Z_{p+1,q}$  is much easier to understand than  $Y_{p,q}$ . For example, it is a simplicial complex, as opposed to  $Y_{p,q}$ . It is in fact it is the simplicial realisation  $|\mathcal{Z}_{p+1,q}|$  of the poset  $(\mathcal{Z}_{p+1,q}, \leq)$  with

$$\mathcal{Z}_{p+1,q} = \{HB \mid A < B \text{ elementary expansion}\},$$

where  $\leq$  is given by

$$HB_0 \leq HB_1 \iff \text{there is some } h \in H \text{ with } B_0 \leq hB_1.$$

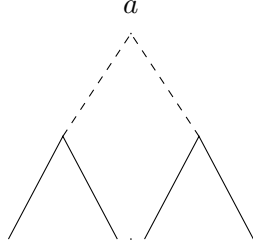
We let

$$\hat{N} = \prod_{i=1}^s (n_i + 1).$$

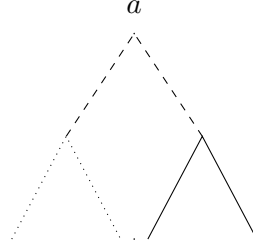
**Proposition 2.7.** *Let  $U_r(\Sigma)$  be a complete, valid, and bounded Cantor algebra. If  $q - p \geq \hat{N} - 2^s$ , and  $p \geq 2^s$ , then  $Z_{p+1,q}$  is contractible.*

*Proof.* Recall that we may view  $Z_{p+1,q}$  as the geometrical realisation of the poset  $(\mathcal{Z}_{p+1,q}, \leq)$ .

We say that  $A < B$  is a *full* extension if the following property holds: for each  $a \in A$ , the set of colours in the path from  $a$  to each of its descendants in  $B$  is the same, i.e., all such paths have the same length. For example, if we had three colours denoted by bold, dashed, and dotted lines, then:



is a full extension;



is not a full extension.

We also say that  $HB \in \mathcal{Z}_{p+1,q}$  is *full* if  $A < B$  is a full extension. Note that this does not depend on the chosen representative in the coset  $HB$ . These elements form a subposet of  $\mathcal{Z}_{p+1,q}$  which we denote by  $\mathcal{Z}_{p+1,q}^f$ . Now, we claim that for any  $HB \in \mathcal{Z}_{p+1,q}$  there is an  $A_0$  with  $HA_0 \in \mathcal{Z}_{p+1,q}^f$ , such that for any other  $HC < HB$  with  $HC \in \mathcal{Z}_{p+1,q}^f$  we have  $HC \leq HA_0 < HB$ .

To see this, consider the expansion  $A < B$  and for each  $a \in A$  let  $\Omega_a$  be the largest set of colours which appear in the path from  $a$  to each of its descendants in  $B$ . Note that unless  $a$  is unchanged in  $B$ , the set  $\Omega_a$  contains at least one element. Let  $A_0$  be obtained from  $A$  by applying to every  $a \in A$  precisely the operations  $\Omega_a$  in such a way that  $A < A_0$  is complete. Then obviously  $A < A_0 < B$  and  $A \neq A_0$ ; thus  $HA_0 \in \mathcal{Z}_{p+1,q}^f$ . Now if  $HC < HB$  is full, by changing  $C$  into  $hC$  for some  $h \in H$  if necessary, we may assume that  $A < C < B$  and that  $A < C$  is full. But the way  $A_0$  is defined together with the fact that  $A < B$  is elementary (i.e., no repetitions of colours are allowed in any path), imply that  $A_0 \geq C$ .

Now using Quillen's Poset Lemma (see [1, Lemma 6.5.2]) we deduce that the poset inclusion  $\mathcal{Z}_{p+1,q}^f \hookrightarrow \mathcal{Z}_{p+1,q}$  induces a homotopy equivalence

$$|\mathcal{Z}_{p+1,q}^f| \simeq |\mathcal{Z}_{p+1,q}| = Z_{p+1,q}.$$

We want to further reduce the poset  $\mathcal{Z}_{p+1,q}^f$ . Observe first that each full extension  $A < B$  is uniquely determined by the  $p$ -tuple  $\omega_B = (\Omega_a)_{a \in A}$  of subsets  $\Omega_a \subseteq S$  needed to get from  $a$  its descendants in  $B$ . As  $H$  permutes the elements of  $A$ , the elements of  $\mathcal{Z}_{p+1,q}^f$  are  $H$ -orbits of tuples of the previous form. Let  $\mathcal{Z}_{p+1,q}^{fn}$  be the subposet of  $\mathcal{Z}_{p+1,q}^f$  consisting of orbits for which  $\Omega_a = \Omega_b$  for  $a \neq b$  both in  $A$  implies  $\Omega_a = \Omega_b = \emptyset$ . Obviously this property does not depend on the chosen representative. As before we have the poset inclusion  $\mathcal{Z}_{p+1,q}^{fn} \hookrightarrow \mathcal{Z}_{p+1,q}^f$ , and for any full extension  $A < B$  there is some  $A < A_0 \leq B$  with  $HA_0 \in \mathcal{Z}_{p+1,q}^{fn}$  and such that, for any  $A < C < B$  with  $HC \in \mathcal{Z}_{p+1,q}^{fn}$ , we have  $HC \leq HA_0$ . To see it note that it suffices to let  $A_0$  be the expansion corresponding to the  $p$ -tuple obtained from  $\omega_B$  by changing all repeated instances of  $\Omega_a$ 's except one to  $\emptyset$ . Also note that it does not matter which element  $a$  associated to the particular instance that is unchanged, as with any other choice we would end up in another representative of the same orbit.

Now, using Quillen's Poset Lemma again, we get a homotopy equivalence

$$|\mathcal{Z}_{p+1,q}^{fn}| \simeq |\mathcal{Z}_{p+1,q}^f| \simeq |\mathcal{Z}_{p+1,q}| = Z_{p+1,q}.$$

Now, the elements of the smaller poset  $\mathcal{Z}_{p+1,q}^{fn}$  can be seen as sets of the form

$$\alpha = \{\Omega_1, \dots, \Omega_m \mid m \leq p, \emptyset \neq \Omega_i \subseteq S\}$$

Recall that repetitions are not allowed. Therefore,  $\mathcal{Z}_{p+1,q}^{fn}$  is a subposet of the poset  $\mathcal{A}$  whose elements are the  $\alpha$ 's above, and given two elements

$$\begin{aligned} \alpha &= \{\Omega_1, \dots, \Omega_m \mid m \leq p, \emptyset \neq \Omega_i \subseteq S\}, \\ \alpha' &= \{\Omega'_1, \dots, \Omega'_{m'} \mid m' \leq p, \emptyset \neq \Omega'_i \subseteq S\}, \end{aligned}$$

we have  $\alpha' \leq \alpha$  if and only if  $m' \leq m$  and there is some reordering of the components of  $\alpha'$  such that  $\Omega'_i \subseteq \Omega_i$  for  $i = 1, \dots, m'$ . Obviously,  $\mathcal{A}$  is contractible as

$$\beta = \{\Omega \mid \emptyset \neq \Omega \subseteq S\}$$

is an upper bound for any other element  $\alpha \in \mathcal{A}$  (the fact that  $\beta \in \mathcal{A}$  is guaranteed by the hypothesis  $p \geq 2^s$ ). So we only have to check that the hypothesis on  $q - p$  implies  $\beta \in \mathcal{Z}_{p+1,q}^{fn}$ . To do that, observe that the cardinality of  $\beta$  is  $2^s - 1 \leq p$ , so all the expansions associated to  $\beta$  can be performed in  $A$ . For each element  $a \in A$ , performing the descending operations encoded by  $\Omega = \{s_{i_1}, \dots, s_{i_d}\}$  ( $i_1, \dots, i_d \in S$ ) yields exactly  $N_\Omega = n_{i_1} \cdots n_{i_d}$  elements. Therefore, to be able to perform the expansions encoded by  $\beta$  inside  $\mathcal{Z}_{p+1,q}^{fn}$ , we need

$$q - p \geq \sum \{N_\Omega - 1 \mid \Omega \in \beta\} = \sum \{N_\Omega \mid \Omega \in \beta\} - 2^s = \hat{N} - 2^s.$$

As this holds true by hypothesis, we get the result.  $\square$

As seen in the proof of Proposition 2.7, to compute the reduced homology of  $Z_{p+1,q}$  we need to consider only  $\mathcal{Z}_{p+1,q}^{fn}$ , which is the geometric realisation of the poset  $\mathcal{Z}_{p+1,q}^{fn}$ . This poset has bounded dimension, with this bound only depending on the number of colours involved. To begin with, suppose  $q - p \geq \hat{N} - 2^s$ . Then  $Z_{p+1,q}$  is contractible and  $\mathcal{Z}_{p+1,q}^{fn}$  has a maximal element  $\beta$ . Consider the poset  $\overline{\mathcal{Z}}_{p+1,q}^{fn} = \mathcal{Z}_{p+1,q}^{fn} \cup \hat{0}$ , where  $\hat{0}$  denotes a unique minimal element. Then  $\overline{\mathcal{Z}}_{p+1,q}^{fn}$  is equal to the interval  $[\hat{0}, \beta]$ , and we can find the length of a maximal chain as follows. We give the colours the natural ordering and begin by including the one-element sets, i.e.  $\hat{0} < \{\{1\}\} < \{\{1\}, \{2\}\} < \dots < \{\{1\}, \{2\}, \dots, \{s\}\}$ . Given any  $2 \leq k \leq s$ , suppose we have  $\alpha \in [\hat{0}, \beta]$  an element of our maximal chain constructed in such a way that, for all  $l < k$ , all  $l$ -element sets  $\Omega_l \in \alpha$ . We now add a  $k$ -element subset  $\Omega_k \notin \alpha$  and get an interval  $[\alpha, \alpha \cup \Omega_k]$  of lengths  $k$  as follows. There is a  $(k-1)$ -element subset  $\Omega_{k-1} \in \alpha$  such that  $\Omega_k = \Omega_{k-1} \cup \{i\}$  for some  $i \in S$ . Then  $\alpha < \{\alpha \setminus \Omega_{k-1}, (\Omega_{k-1} \cup \{i\})\} = \alpha_1$ . Now there is a  $(k-2)$ -element subset  $\Omega_{k-2} \in \alpha_1$  such that  $\Omega_{k-1} = \Omega_{k-2} \cup \{j\}$  for some  $i \neq j \in S$ , and  $\alpha < \alpha_1 < \{\alpha_1 \setminus \Omega_{k-2}, (\Omega_{k-2} \cup \{j\})\} = \alpha_2$ . Continue this process to arrive at

$$\alpha < \alpha_1 < \dots < \alpha_k = \alpha \cup \Omega_k.$$

Continue this process until finally we construct the only  $s$ -element subset  $\Omega_s = S$ . Hence the length of a maximal chain in  $[\hat{0}, \beta]$  is

$$D(s) = \sum_{i=1}^s i \binom{s}{i} = s2^{s-1}.$$

This implies that the dimension of  $Z_{p+1,q}^{fn}$  is equal to  $D(s) - 1$ . Now removing the condition on  $q - p$ , we only decrease the dimension and hence

$$\dim(Z_{p+1,q}^{fn}) \leq D(s) - 1,$$

and  $Z_{p+1,q}^{fn}$  is contractible if  $\dim(Z_{p+1,q}^{fn}) = D(s) - 1$ .

We are now ready to prove Theorem 1.1: if  $U_r(\Sigma)$  is a complete, valid, and bounded Cantor algebra,  $s \geq 2$  and  $k \geq D(s)$ , then  $H_k(V_r(\Sigma), \mathbb{Q}) = 0$ .

*Proof of Theorem 1.1.* Let  $p$  be as in Proposition 2.3 and as in Proposition 2.7 and set  $q = p + tk$  with  $t = N - 1$ . As  $X_{p,q}$  is  $k$ -connected,  $H_k(V_r(\Sigma)) = H_k(Y_{p,q})$ . Consider for  $j \geq 0$  the long exact sequence of Lemma 2.6:

$$\dots \rightarrow H_k(Y_{p+j+1,q}) \rightarrow H_k(Y_{p+j,q}) \rightarrow \tilde{H}_{k-1}(Z_{p+j+1,q}) \rightarrow \dots$$

When  $j = (t-1)k$ , note that  $\dim Y_{p+j+1,q} = p + tk - (p + tk - k + 1) = k - 1$  thus

$$\dots \rightarrow 0 = H_k(Y_{p+(t-1)k+1,q}) \rightarrow H_k(Y_{p+(t-1)k,q}) \rightarrow \tilde{H}_{k-1}(Z_{p+(t-1)k+1,q}) \rightarrow \dots$$

Proposition 2.7 implies that  $\tilde{H}_{k-1}(Z_{p+j+1,q}) \cong \tilde{H}_{k-1}(Z_{p+j+1,q}^{fn})$ . The argument above implies that  $Z_{p+j+1,q}^{fn}$  either has dimension strictly less than  $k - 1$  or is contractible. Hence  $\tilde{H}_{k-1}(Z_{p+j+1,q}) = 0$  for all  $0 \leq j \leq (k-1)t$ . This means that

$$\begin{aligned} 0 = H_k(Y_{p+(t-1)k+1,q}) &\cong H_k(Y_{p+(t-1)k-1,q}) \cong \dots \\ &\dots \cong H_k(Y_{p+1,q}) \cong H_k(Y_{p,q}) \cong H_k(V_r(\Sigma)). \end{aligned}$$

□

**Corollary 2.8.** *The Higman-Thompson groups  $V_{n,r}$  are rationally acyclic.*

*Proof.* Use [5, Theorem 2] instead of Proposition 2.3. The computations for  $D(s)$  also work for  $s = 1$  and the argument of Theorem 1.1 goes through unchanged. □

**Corollary 2.9.** *Let  $sV$  be Brin-Thompson group. Then for all  $k \geq s2^{s-1}$  and  $k = 1$ , we have  $H_k(sV) = 0$ .*

*Proof.* For  $k \geq D(s)$  this is Theorem 1.1 and for  $k = 1$  this follows from the fact that  $sV$  is simple [3]. □

*Remark 2.10.* We do not know whether the bound in Theorem 1.1 is sharp, yet the computations in Theorem 2.11 below show that it is sharp for  $2V$ . It is the same for all groups  $V_r(\Sigma)$  for a given  $s$ , yet it is not clear how much arities influence the homology of the complex  $Z_{p+j+1,q}^{fn}$ . The proof for  $2V$  below uses the fact that all  $n_1 = n_2 = 2$  when considering explicitly the complexes  $Z_{p+j+1,q}^{fn}$ .



**Theorem 2.11.** *Let  $G = 2V$ . Then*

$$H_3(2V) \cong \mathbb{Q},$$

*and for all  $k \neq 3, 0$*

$$H_k(2V) = 0.$$

*Proof.* In light of Corollary 2.9, noting that  $D(2) = 4$ , we only need to verify the claim for  $k = 2, 3$ . We begin by computing the reduced homology of  $Z_{p+1,q}$  for all  $q - p \leq 4$ . Using the description at the end of the proof of Proposition 2.7 we get the following cases:

- $q - p = 1$  : Here  $Z_{p+1,q}$  consists of two points and hence  $\tilde{H}_0(Z_{p+1,q}) \cong \mathbb{Q}$  and  $\tilde{H}_i(Z_{p+1,q}) = 0$  for all  $i \geq 1$ .
- $q - p = 2$  : Here  $Z_{p+1,q} \simeq *$  and  $\tilde{H}_i(Z_{p+1,q}) = 0$  for all  $i \geq 0$ .
- $q - p = 3$  :  $Z_{p+1,q} \simeq S^1$  and  $\tilde{H}_1(Z_{p+1,q}) \cong \mathbb{Q}$  and  $\tilde{H}_i(Z_{p+1,q}) = 0$  for all  $i \neq 1$ .
- $q - p = 4$  :  $Z_{p+1,q} \simeq S^2$  and  $\tilde{H}_2(Z_{p+1,q}) \cong \mathbb{Q}$  and  $\tilde{H}_i(Z_{p+1,q}) = 0$  for all  $i \neq 2$ .

We compute  $H_2(2V)$  : Here  $q = p + 6$  and for  $j \geq 4$  the dimension of  $Y_{p+j+1,q}$  is less or equal to 1 and hence the second homology vanishes.

For  $j = 4$ , we have  $H_2(Y_{p+4+1,q}) = 0$  and  $\tilde{H}_1(Z_{p+4+1,q}) = 0$  because  $Z_{p+4+1,q}$  is contractible. Thus the long exact sequence yields  $H_2(Y_{p+3+1,q}) = 0$ .

For  $j = 3$  we have that  $q - (p + j) = 3$  and hence  $H_1(Z_{p+3+1,q}) \cong \mathbb{Q}$  and we obtain the following long exact sequence:

$$0 \rightarrow H_2(Y_{p+3,q}) \rightarrow \mathbb{Q} \rightarrow H_1(Y_{p+3+1,q}) \rightarrow H_1(Y_{p+3,q}). \quad (*)$$

Proposition 2.3 and Corollary 2.9 imply that  $H_1(Y_{p+3,q}) \cong H_1(2V) = 0$ . Computing  $H_1(Y_{p+3+1,q})$  is the same as computing  $H_1(Y_{p'+1,p'+3})$ . Since  $Y_{p'+3,p'+3}$  is zero-dimensional,  $H_1(Y_{p'+3,p'+3}) = 0$  and  $\tilde{H}_0(Z_{p'+2+1,p'+3}) \cong \mathbb{Q}$ . Furthermore, since  $Y_{p'+3,p'+3}$  is connected, we have  $\tilde{H}_0(Y_{p'+3,p'+3}) = 0$ . Hence the long exact sequence of Lemma 2.6 yields that  $H_1(Y_{p'+2,p'+3}) \cong \mathbb{Q}$ . We now have the long exact sequence:

$$\tilde{H}_1(Z_{p'+1+1,p'+3}) \rightarrow H_1(Y_{p'+2,p'+3}) \rightarrow H_1(Y_{p'+1,p'+3}) \rightarrow \tilde{H}_0(Z_{p'+1+1,p'+3}).$$

Since  $Z_{p'+1+1,p'+3}$  is contractible, the two outside terms are zero and hence  $H_1(Y_{p'+1,p'+3}) \cong \mathbb{Q}$ . This means that  $(*)$  reduces to

$$0 \rightarrow H_2(Y_{p+3,q}) \rightarrow \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow 0.$$

Since the map  $\mathbb{Q} \rightarrow \mathbb{Q}$  is onto, it is an isomorphism and this implies that  $H_2(Y_{p+3,q}) = 0$ .

For  $j = 2$  we get that  $H_1(Z_{p+2+1,q}) = 0$ . Hence the long-exact sequence of Lemma 2.6 yields

$$H_2(Y_{p+2,q}) = 0.$$

Now, for  $j \leq 1$ ,  $Z_{p+j+1,q}$  is contractible and as above we have that

$$H_2(2V) \cong H_2(Y_{p,q}) = 0.$$

We finish by computing  $H_3(2V)$ , where  $q = p + 9$ . For  $j \geq 6$  we have  $\dim(Y_{p+j+1,q}) \leq 2$  and hence  $H_3(Y_{p+6+1,q}) = 0$ . Now we repeatedly use the long exact sequence of Lemma 2.6.

For  $j = 6$  we have that  $q - (p + j) = 3$  and hence  $H_3(Z_{p+6+1,q}) = H_2(Z_{p+6+1,q}) = 0$  and

$$H_3(Y_{p+6+1}) = H_3(Y_{p+6,q}) = 0.$$

Let  $j = 5$ . Then  $H_3(Z_{p+5+1,q}) = 0$  and  $H_2(Z_{p+5+1,q}) \cong \mathbb{Q}$ . Furthermore  $H_2(Y_{p+5+1,q}) = 0$ , which follows from the calculations for  $H_2(2V)$ , where we showed that  $H_2(Y_{p',p'+3}) = 0$ . Hence the long exact sequence now gives

$$H_3(Y_{p+5,q}) \cong H_2(Z_{p+5+1,q}) \cong \mathbb{Q}.$$

Now, for  $j \leq 4$ , we have that  $Z_{p+j+1,q}$  is contractible, hence the second and third homology vanishes, and hence  $H_3(Y_{p+j+1,q}) \cong H_3(Y_{p+j,q}) \cong \mathbb{Q}$ , which yields

$$H_3(2V) \cong \mathbb{Q}. \quad \square$$

When computing  $H_2(2V)$  above, we used the fact that  $2V$  was simple. One can also get the result without referring to the simplicity of  $2V$  by explicitly computing the connecting homomorphism in  $(*)$ . One could then continue by using the long exact sequence of Lemma 2.6 to show that  $H_1(2V) = 0$ .

### 3. CONJUGACY CLASSES OF FINITE CYCLIC SUBGROUPS IN $sV_{n,r}$

**Proposition 3.1.** *For any  $m > 0$  such that  $m$  and  $n - 1$  are coprime, the number of conjugacy classes of cyclic groups of order  $m$  in  $sV_{n,r}$  is*

$$n^{\phi(m)} - 1,$$

where  $\phi(m)$  is the Euler function. In particular, for any  $m > 0$  the number of conjugacy classes of cyclic groups of order  $m$  in  $sV_{2,r}$  is

$$2^{\phi(m)} - 1.$$

*Proof.* We claim first that the number of conjugacy classes of cyclic subgroups of order  $m$  equals the cardinality of the set  $C(n, r, m)$  of the possible

$$\{k_s \mid s \mid m, 0 \leq k_s \leq n - 1\}$$

such that  $k_m \neq 0$  and  $\sum_{s \mid m} s k_s \equiv r \pmod{n - 1}$ . The claim follows essentially from [13, Proposition 4.2 and Theorem 4.3], but we briefly recall the argument here. The main idea goes back to Higman (see [10] and also [7]): one can prove that any finite subgroup  $H \leq sV_{n,r}$  acts on a certain  $n$ -ary  $r$ -rooted forest by permuting the set of leaves  $L$ . That set of leaves can be split into transitive subsets, each corresponding to a particular type as a permutation representation, and at least one of the orbits must be faithful. The number of orbits of a certain type can be modified modulo  $n - 1$  just by passing to subtrees. This yields the same group, yet note that if some type does not appear it can not be created. Any other copy  $H_1 \leq sV_{n,r}$  of the same group is conjugate to  $H$  in  $sV_{n,r}$  if and only if, in both groups for each type, the set of orbits of that precise type is either zero in both, or non zero in both and congruent modulo  $n - 1$ .

In the particular case of a cyclic group of order  $m$ , the types of permutation representations correspond to the possible divisors of  $m$ , more precisely, to each  $s \mid m$  we may associate the transitive permutation representation of length  $s$ . The number  $k_s$  above refers to the number of orbits of length  $s$ .

The only faithful orbit has length  $m$ , hence  $k_m \neq 0$ . From the fact that our group is defined using an  $r$ -rooted forest we get  $\sum_{s|m} s k_s \equiv r \pmod{n-1}$ .

Now, assume that  $m$  and  $n-1$  are coprime. Then for each choice of  $k_s$ ,  $s|m$ ,  $s \neq 1$ , solving the equation

$$mx \equiv r - \sum_{s|m, s \neq 1} \frac{m}{s} k_s \pmod{n-1}$$

yields exactly one possible value of  $k_1$ , i.e., a single element in the set  $C(n, r, m)$  above. Note here that we are using the fact that  $k_1$  can not be zero, otherwise there could in some cases be two valid choices for  $k_1$ , namely 0 and  $n-1$ . This means that

$$|C(n, r, m)| = n^{\phi(m)} - 1. \quad \square$$

For any  $m > 0$  the number of conjugacy classes of cyclic groups of order  $m$  in  $sV_{n,r}$  can be computed as follows: Consider the set

$$\Omega = \{r - \sum_{s|m, 0 < s \leq m} \alpha(s)s \mid 0 \leq \alpha(s) \leq n\} \subseteq \mathbb{Z},$$

observe that it consist of  $n^{\phi(m)-1}$  elements. Let  $d = \gcd(m, n-1)$ , the desired number is then

$$d \cdot |\{a \in \Omega \mid d \text{ divides } a\}|.$$

More detailed formulas for the special case when  $m = p^a$  for a prime  $p$  can be found in [10] or [7].

#### 4. ASSEMBLY MAPS AND ALGEBRAIC K-THEORY

In this final section we explain an application of our computations to algebraic  $K$ -theory. We begin by recalling the rationalised version of the Farrell-Jones Conjecture. For more details and background information we refer to the introduction of [12] and to the references cited there.

Let  $G$  be a group. Denote by  $(\mathcal{FC})$  the set of conjugacy classes of finite cyclic subgroups of  $G$ . The centraliser of a subgroup  $C \leq G$  is denoted  $Z_G(C)$  and the Weyl group  $W_G(C)$  is defined as the quotient  $W_G(C) = N_G(C)/Z_G(C)$  of the normaliser modulo the centraliser. Notice that, when  $C$  is finite, then  $W_G(C)$  is finite, too.

For any group  $G$  and any  $n \in \mathbb{Z}$  there is a natural homomorphism

$$(4.1) \quad \bigoplus_{(C) \in (\mathcal{FC})} \bigoplus_{\substack{p+q=n \\ p \geq 0 \\ q \geq -1}} H_p(Z_G(C); \mathbb{Q}) \otimes_{\mathbb{Q}[W_G(C)]} \Theta_C \left( K_q(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right) \rightarrow K_n(\mathbb{Z}[G]) \otimes_{\mathbb{Z}} \mathbb{Q},$$

called *rationalised Farrell-Jones assembly map*. The *rationalised Farrell-Jones Conjecture* asserts that (4.1) is an isomorphism for every  $G$  and every  $n \in \mathbb{Z}$ .

Here  $\Theta_C$  is an idempotent endomorphism of  $K_q(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}$ , whose image is a direct summand of  $K_q(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q}$  isomorphic to

$$(4.2) \quad \text{coker} \left( \bigoplus_{D \not\leq C} K_q(\mathbb{Z}[D]) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_q(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q} \right).$$

The Weyl group acts via conjugation on  $C$  and hence on  $\Theta_C(K_q(\mathbb{Z}[C]) \otimes_{\mathbb{Z}} \mathbb{Q})$ . The Weyl group action on the homology comes from the fact that the space  $EN_G(C)/Z_G(C)$  is a model for  $BZ_G(C)$ . The dimensions of the  $\mathbb{Q}$ -vector spaces in (4.2) are explicitly computed in [15]\*Theorem on page 9 for any  $q$  and any finite cyclic group  $C$ .

The following injectivity result about the Farrell-Jones Conjecture is proven in [12].

**Theorem 4.3** ([12]\*Theorem 1.11). *Suppose that the following conditions are satisfied for each finite cyclic subgroup  $C$  of  $G$ .*

- [A] *For every  $p \geq 1$ ,  $H_p(Z_G(C); \mathbb{Z})$  is a finitely generated abelian group.*
- [B] *For every  $q \geq 0$ , the natural homomorphism*

$$K_q(\mathbb{Z}[\zeta_c]) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \prod_{\ell \text{ prime}} K_t(\mathbb{Z}_{\ell} \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_c], \mathbb{Z}_{\ell}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

*is injective, where  $c$  is the order of  $C$  and  $\zeta_c$  is any primitive  $c$ -th root of unity.*

*Then the restriction of the rationalised Farrell-Jones assembly map (4.1) to the summands with  $q \geq 0$  is injective for each  $n$ .*

We remark that condition [B] is conjecturally always true; more precisely, it is true if the Leopoldt-Schneider Conjecture in algebraic number theory holds for all cyclotomic fields. For more details we refer to [12, Section 2].

We can now prove Theorem 1.2.

*Proof of Theorem 1.2.* First, we need to show that the source of the rationalised Farrell-Jones assembly map (4.1) is isomorphic to (1.3):

$$\bigoplus_{1 \leq m} \bigoplus_{1 \leq i \leq 2^{\phi(m)} - 1} \bigoplus_{\substack{p+q=n \\ 0 \leq p \leq \phi(m)s2^{s-1} \\ -1 \leq q}} H_p(Z_G(C_m^i); \mathbb{Q}) \otimes_{\mathbb{Q}[W_G(C_m^i)]} \Theta_{C_m^i} \left( K_q(\mathbb{Z}[C_m^i]) \otimes_{\mathbb{Z}} \mathbb{Q} \right).$$

Using Proposition 3.1 we obtain the number of conjugacy classes of finite cyclic subgroups of order  $m$ . It was shown in [13, Theorem 4.4] that each centraliser is decomposed into a direct product  $Z_{sV}(C_m) = Z_1 \times \cdots \times Z_l$ , where  $l$  is the number of transitive permutation representations of  $C_m$ , and where each  $Z_i$  fits into a short exact sequence of groups

$$0 \rightarrow K_i \rightarrow Z_i \rightarrow sV \rightarrow 0$$

with  $K_i$  a locally finite group. As already observed in the proof of Proposition 3.1,  $l = \phi(m)$ . Now an easy spectral sequence argument shows that for all  $p \geq 0$

$$H_p(Z_i; \mathbb{Q}) \cong H_p(sV; \mathbb{Q}).$$

The Künneth Theorem together with Corollary 2.9 yields that, if  $p \geq \phi(m)s2^{s-1}$ , then  $H_p(Z_{sV}(C_m); \mathbb{Q})$  vanishes. This establishes the first statement of the theorem.

The last statement is then a consequence of Theorem 4.3, because [14, Theorems 3.1 and 4.9] imply that  $sV$  and all centralisers of finite subgroups are of type  $F_{\infty}$ , and therefore condition [A] is satisfied.  $\square$

*Remark 4.4.* The Weyl groups  $W_{sV}(C)$  for finite subgroups of  $sV$  were described in [14, Theorem 5.1]. With the notation used in the proof of Proposition 3.1, for each finite cyclic group of order  $m$  there is a set of leaves  $Y$  of a fixed order  $|Y| = \sum_{s|m} sk_s$ , where  $0 \leq k_s \leq 1$  and  $k_m = 1$ . Note that here  $n = 2$ . Hence, for each representative  $C_m^i$  of the conjugacy classes of cyclic subgroups of order  $m$ , there is a  $Y_i$  with  $m \leq |Y_i| \leq \sum_{s|m} s$  and

$$W_{sV}(C_m^i) = N_{S(Y_i)}(C_m^i) / Z_{S(Y_i)}(C_m^i),$$

where  $S(Y_i)$  denotes the symmetric group on  $|Y_i|$  letters.

Notice that, in the case of Thompson's group  $V$ , the formula (1.3) drastically simplifies, since  $H_p(V; \mathbb{Q})$  vanishes for all  $p \neq 0$ , and so (1.3) reduces to

$$\bigoplus_{1 \leq m} \bigoplus_{1 \leq i \leq 2^{\phi(m)} - 1} \mathbb{Q} \otimes_{\mathbb{Q}[W_G(C_m^i)]} \Theta_{C_m^i} \left( K_n(\mathbb{Z}[C_m^i]) \otimes_{\mathbb{Z}} \mathbb{Q} \right).$$

Also for Brin-Thompson's group  $2V$ , using Theorem 2.11, one can describe (1.3) in a little more detail.

Finally, we remark that [12, Theorem 1.16] applies to general automorphism groups of Cantor algebras because of [14, Theorems 3.1 and 4.9], and immediately gives the following result about their Whitehead groups.

**Corollary 4.5.** *Let  $G = V_r(\Sigma)$  be the automorphism group of a valid, bounded, and complete Cantor algebra. Then there is an injective homomorphism*

$$\varinjlim_{\mathcal{O}_{\mathcal{F}} G} Wh(H) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow Wh(G) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where the colimit is taken over the orbit category of finite subgroups of  $G$ . In particular,  $Wh(G) \otimes_{\mathbb{Z}} \mathbb{Q}$  is an infinite dimensional  $\mathbb{Q}$ -vector space.

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